Video 9

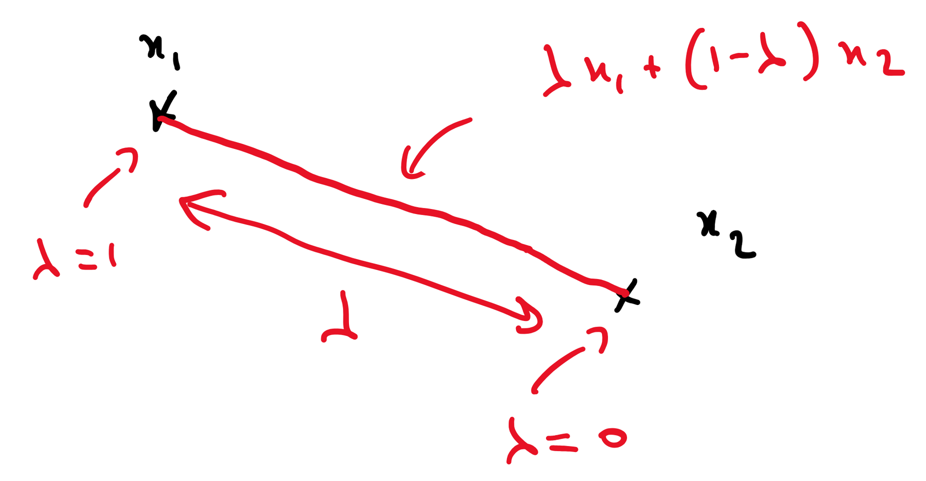
Sunday, October 4, 2020 2:06 PM

How do we prove that XOR is not linearly separable?

↳ here, we use convexity

Convexity A set S is convex if any line segment connecting points in S lies entirely within S.

$$x_1, x_2 \in S \implies \lambda x_1 + (1-\lambda)x_2 \in S \quad \forall\, 0 \le \lambda \le 1$$



$x_1, \dots, x_N \in S$, all weighted averages (i.e., convex combinations) also lie within the set S.

$$\lambda_1 x_1 + \dots + \lambda_N x_N \in S \quad \text{for } \lambda_i > 0 \ \& \ \sum_{i=1}^{N} \lambda_i = 1$$

Proof by induction:

$N=2$ $\quad \sum_{i=1}^{2} \lambda_i = 1$, $\lambda_1 = 1 - \lambda_2$

By definition $\lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 x_1 + (1-\lambda_1)x_2 \in S$

Let's assume that the property holds for $N-1$

Let's prove it holds for N

$$\sum_{i=1}^{N} \lambda_i x_i = \underbrace{\lambda_N x_N}_{\in S} + \underbrace{\sum_{i=1}^{N-1} \lambda_i x_i}_{\text{divide each term of the sum by } 1-\lambda_N} = \lambda_N x_N + (1-\lambda_N) \underbrace{\sum_{i=1}^{N-1} \frac{\lambda_i}{1-\lambda_N} x_i}_{\text{goal: Show that this is also in } S}$$

- Show that $\forall i \in 1..N-1$, $\dfrac{\lambda_i}{1-\lambda_N} > 0$

  True because $\lambda_i > 0$ and $1 - \lambda_N > 0$

- $$\sum_{i=1}^{N-1} \frac{\lambda_i}{1-\lambda_N} = \frac{1}{1-\lambda_N} \sum_{i=1}^{N-1} \lambda_i$$

  $$\sum_{i=1}^{N} \lambda_i = 1 = \lambda_N + \sum_{i=1}^{N-1} \lambda_i$$

  $$= \frac{1}{1-\lambda_N}(1-\lambda_N)$$

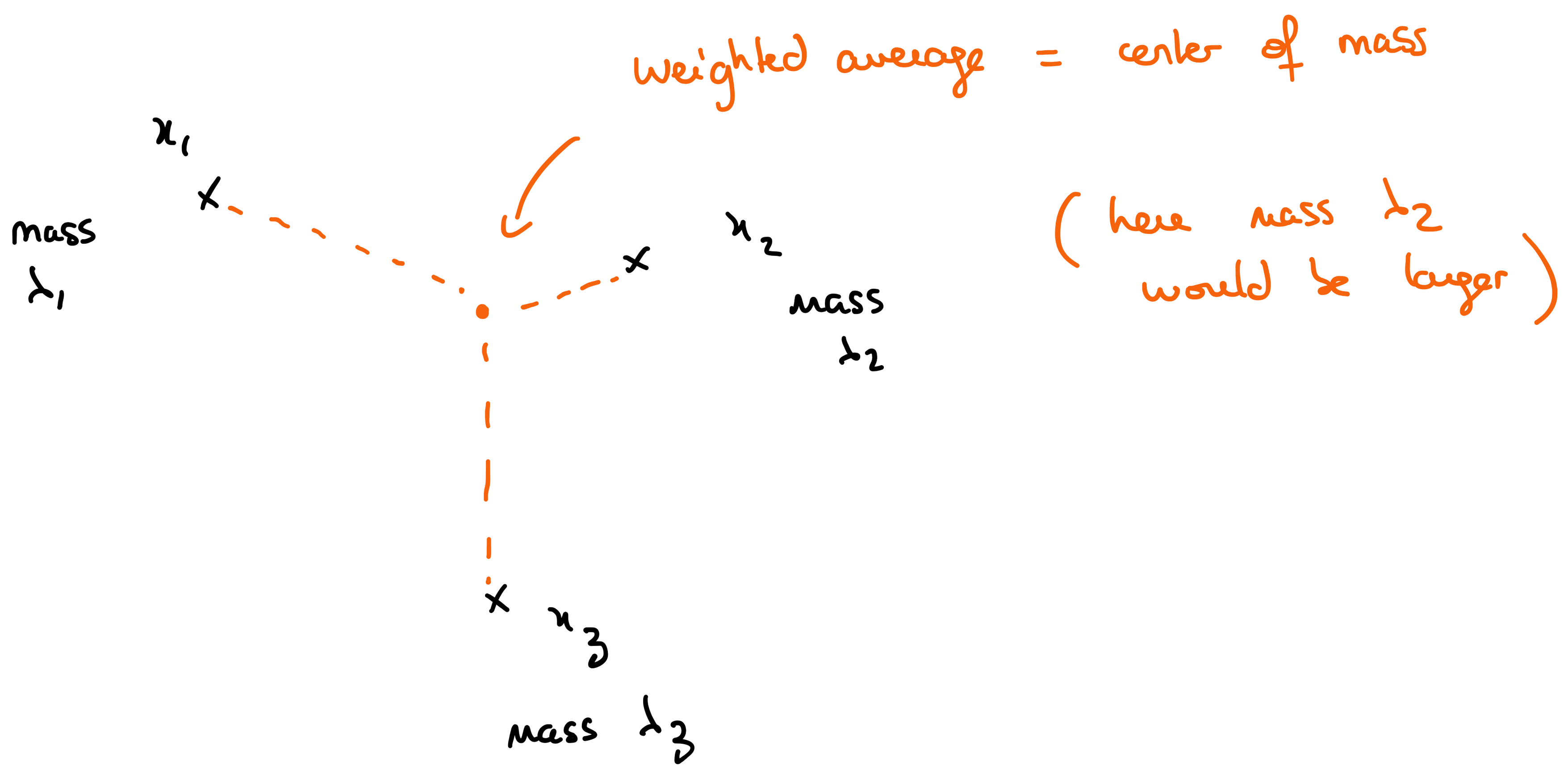
  $$= 1$$

We can deduce that $\displaystyle\sum_{i=1}^{N-1} \frac{\lambda_i}{1-\lambda_N} x_i \in S$

We have: $$\sum_{i=1}^{N} \lambda_i x_i = \underbrace{\underbrace{\lambda_N x_N}_{\in S} + \underbrace{(1-\lambda_N)}_{\lambda'} \underbrace{\sum_{i=1}^{N-1} \frac{\lambda_i}{1-\lambda_N} x_i}_{\in S}}_{\in S}$$

$$\lambda_N + \lambda' = \lambda_N + 1 - \lambda_N = 1$$

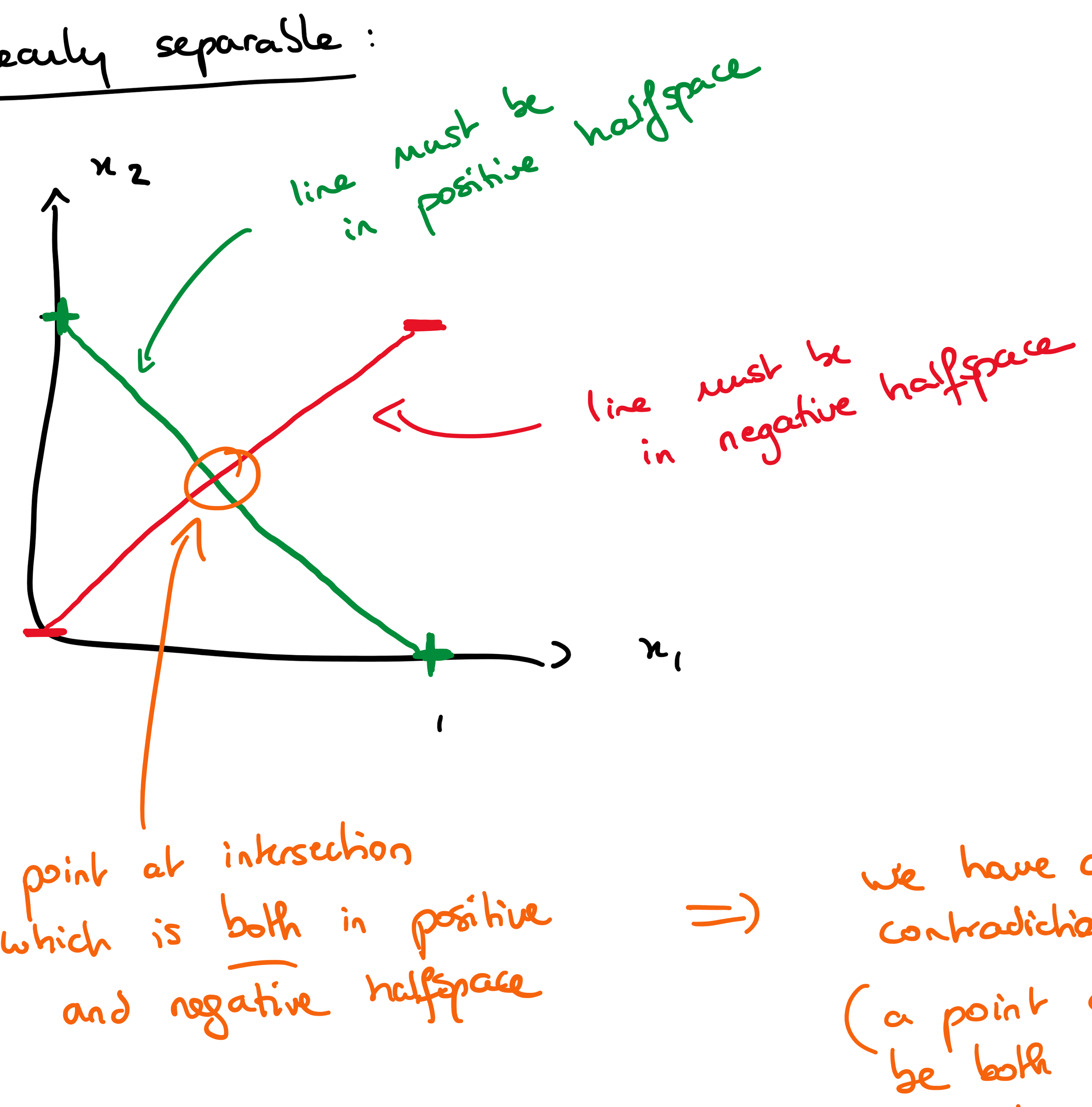
By induction, we conclude the property holds for $N > 2$



In binary classification, 2 sets are always convex:

- In data/input space: both positive and negative regions are convex.
- in weight space: feasible region is convex.

XOR is not linearly separable:



$\implies$ we have a contradiction (a point cannot be both positive and negative)