# Lagrange Multiplier

## Constrained Optimization

Suppose that we would like to solve the following optimization problem

$$\min_{\bar{x}} f_0(\bar{x})$$
$$\text{subject to } f_i(\bar{x}) \leq 0, i = 1, \cdots, m. \tag{1}$$

Let $p^*$ be a solution of (1).
Define $\mathcal{L}(\bar{x}, \bar{\lambda})$ where $\bar{\lambda} = (\lambda_1, \cdots, \lambda_m)$ and $\lambda_i \geq 0$ as

$$\mathcal{L}(\bar{x}, \lambda) := f_o(\bar{x}) + \sum_{i=1}^{m} \lambda_i f_i(\bar{x}). \tag{2}$$

Note that

$$p^* = \min_{\bar{x}} \max_{\bar{\lambda}} \mathcal{L}(\bar{x}, \bar{\lambda})$$

Also, the function $\min_{\bar{x}} \mathcal{L}(\bar{x}, \bar{\lambda})$ is a concave function, so it always has a maximum. Let $d^*$ be this maximum.

$$d^* = \max_{\bar{\lambda}} \min_{\bar{x}} \mathcal{L}(\bar{x}, \bar{\lambda})$$

In general, $d^* \leq p^*$. To see this, let $\phi(x, y)$ be a multi variable function. Then for all $x, y$, we have

$$\phi(x, y) \leq \max_{x} \phi(x, y).$$

Since this is true for all $y$, then

$$\phi(x, y) \leq \min_{y} \max_{x} \phi(x, y).$$

Similarly, for all $x, y$, $\phi(x, y) \geq \min_y \phi(x, y)$ and therefore,

$$\phi(x, y) \geq \max_{x} \min_{y} \phi(x, y).$$

Putting things together, we get

$$\max_{x} \min_{y} \phi(x, y) \leq \min_{y} \max_{x} \phi(x, y).$$

Sometimes, the above inequality is achieved with an equality; for example, if Slater's condition holds. In practice, we prefer solving for $d^*$ rather than $p^*$ because it is easier to deal with.

### Slater's Condition

If there exists a point $\bar{x} \in \mathcal{D}$, where $\mathcal{D} = \bigcap_{i=0}^{m} \mathrm{dom} f_i(\bar{x})$ such that $f_i(\bar{x}) < 0$ for all $i = 1, \cdots, m$, then $d^* = p^*$.

So, to proceed and solve for $d^*$:

1. Solve $\phi(\bar{\lambda}) := \min_{\bar{x}} \mathcal{L}(\bar{x}, \bar{\lambda})$
2. Solve $d^* = \max_{\bar{\lambda}} \phi(\bar{\lambda})$.

**Example**

$$\text{minimize } x^2 + y^2 + 2z^2$$
$$\text{subject to } 2x + 2y - 4z \geq 8.$$

**Solution:**

First, observe that the global minimizer $(0, 0, 0)$ does not satisfy the constraint, so it is not a feasible solution. Next, we define the Lagrangian $\mathcal{L}(x, y, z, \lambda) = x^2 + y^2 + 2z^2 + \lambda(8 - 2x - 2y + 4z)$.

1. Minimize $\mathcal{L}(x, y, z, \lambda)$ with respect to $(x, y, z)$:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \implies 2x - 2\lambda = 0 \implies x = \lambda$$
$$\frac{\partial \mathcal{L}}{\partial y} = 0 \implies 2y - 2\lambda = 0 \implies y = \lambda$$
$$\frac{\partial \mathcal{L}}{\partial z} = 0 \implies 4x + 4\lambda = 0 \implies z = -\lambda.$$

$\phi(\lambda) = \min_{x,y,z} \mathcal{L}(x, y, z, \lambda) = \mathcal{L}(\lambda, \lambda, -\lambda, \lambda) = \lambda^2 + \lambda^2 + 2\lambda^2 + \lambda(8 - 2\lambda - 2\lambda + 4\lambda) = 8\lambda - 4\lambda^2.$

2. Solve for $d^*$

$$\frac{d\phi(\lambda)}{d\lambda} = 0 \implies 8 - 8\lambda = 0 \implies \lambda = 1.$$

$p^* = d^* = \mathcal{L}(1, 1, -1, 1) = 4$ and it is achieved at $(x, y, z) = (1, 1, -1)$.

**Note** Maximums and minimums may not always exist. Therefore, it is preferable to use supremums and infimums in their place where possible. But we have avoided that in favor of an easier notation. All the relations we derived today still hold when we swap out max(.) for sup(.) and inf(.) for min(.). See Chapter 5 of [BV04] for more detailed derivations.

# References

[BV04] Stephen P. Boyd and Lieven Vandenberghe. *Convex optimization.* Cambridge, UK ; New York: Cambridge University Press, 2004, pp. 215–238. ISBN: 978-0-521-83378-3.